THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra (Term 1, 2023-24)
Midterm Test
1st November, 2023

• Write your Name and Student ID on the front page.

• Give full explanation and justification for **all** your calculations and observations, and write all your proofs in a clear and rigorous way.

• Answer all questions. Total score: 100 pts. Time allowed: 90 minutes.

(1) (10 pts) Mark each of the following statements "T / True" (meaning that it is a true statement) or "F / False" (meaning that there are counterexamples to the statement). No reasoning is required. 2 pts for each correct answer; no pts will be deducted for a wrong answer.

(a) A free abelian group is a free group.

(b) Let n be a positive integer. The abelianization of a free group on n generators is a free abelian group on n generators.

(c) Up to isomorphisms, there are 10 abelian groups of order 720.

(d) The commutator subgroup $[G, G]$ of a group G is the biggest normal subgroup of G such that the quotient of G by it is abelian.

(e) For $n \geq 5$, the alternating group A_n has no nontrivial proper subgroups.

Answer and Explanation.
(a) F / False. A free abelian group is not necessarily a free group. Free groups do not have to abide by the commutative law, while free abelian groups do.
(b) T / True. The abelianization of a free group on n generators is indeed a free abelian group on n generators.
(c) T / True. The prime factorization of 720 is $2^4 \times 3^2 \times 5$. The number of abelian groups of order n is the number of partitions of the exponents in the prime factorization of n. The partitions of 4 are (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, and $(1, 1, 1, 1)$. The partitions of 2 are (2) and $(1, 1)$. So there are $5 \times 2 = 10$ abelian groups of order 720.
(d) F / False. The commutator subgroup $[G, G]$ is the smallest normal subgroup of G such that $G/[G, G]$ is abelian. The largest normal subgroup of G such that $G/[G, G]$ is abelian is G itself.

(e) F / False. For $n \geq 5$, the alternating group A_n does have nontrivial proper subgroups. For example, for $n = 5$, A_5 has a subgroup isomorphic to the symmetric group A_3. They do not have nontrivial proper normal subgroups.

(2) (15 pts) Let G be a group and $H \leq G$ be a subgroup of G. Define

$$N := \bigcap_{g \in G} gHg^{-1}.$$

(a) Show that N is a normal subgroup in G.

(b) Show that N is the largest subgroup in H which is normal in G.

Proof. (a) N is a subgroup of G because it is an intersection of subgroups gHg^{-1}.
To prove N is normal, for any $a \in G$, we need to show that $aNa^{-1} = N$.
We know that $N = \bigcap_{g \in G} gHg^{-1}$. Conjugating this equation by a, we get

$$aNa^{-1} = \bigcap_{g \in G} a(gHg^{-1})a^{-1} = \bigcap_{g \in G} (ag)H(ag)^{-1}.$$

As g ranges over all elements in G, so does ag, since left multiplication by a is a bijection from G to itself. Therefore, the set of all $(ag)H(ag)^{-1}$ as g ranges over all of G is the same as the set of all gHg^{-1}. Thus, we obtain $aNa^{-1} = N$, showing that N is normal in G.

(b) Suppose K is a subgroup of H that is normal in G. For any $g \in G$ and $k \in K$, we have $K = gKg^{-1} \subseteq gHg^{-1}$. Therefore,

$$K \subseteq N = \bigcap_{g \in G} gHg^{-1}.$$

proving N is the largest subgroup in H normal in G.

□

(3) (15 pts) The **quaternion group** $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is defined by the relations $i^2 = j^2 = k^2 = ijk = -1$ and $(-1)^2 = 1$. You can use the fact that Q is a group without proof. Find a composition series for Q. Give full proofs to your assertions.

Proof. First, we observe that $G_1 = \langle i \rangle = \{1, i, -1, -i\}$ is a subgroup of Q of order 4. We note that the index $[Q : G_1] = 2$. Therefore, G_1 is a normal subgroup of Q

Next, we consider $G_2 = \langle -1 \rangle = \{1, -1\}$, which is a subgroup of G_1 of order 2. Again, the index $[G_1 : G_2] = 2$ and so G_2 is a normal subgroup of G_1.

Therefore, we have a composition series $1 \trianglelefteq G_2 \trianglelefteq G_1 \trianglelefteq Q$ where each factor is a simple group: $G_2/1 \cong \mathbb{Z}_2$, $G_1/G_2 \cong \mathbb{Z}_2$, and $Q/G_1 \cong \mathbb{Z}_2$. This completes the proof. □

(4) (15 pts) Let G be a group. Let H, N be normal subgroups of G such that $G = HN$ and $H \cap N = \{e\}$.

(a) Show that $G/N \cong H$ and $G/H \cong N$.

(b) Is it true that $G \cong H \times N$? Give full justifications to your assertions.

Proof. (a) To show that $G/N \cong H$, we first note that since N is normal in G, we have a natural projection $\pi : G \to G/N$. We restrict this map to H to obtain a group homomorphism $\phi : H \to G/N$.
The kernel of ϕ is $\ker(\phi) = \{h \in H : \phi(h) = N\} = \{h \in H : h \in N\} = H \cap N = \{e\}$.
To show that ϕ is surjective, we note that each coset $gN \in G/N$ is in the image of ϕ: Given $g \in G$, we can write $g = hn$ for some $h \in H$ and $n \in N$ because $G = HN$. Then $\phi(h) = hN = hnN = gN$.
Therefore, ϕ is an isomorphism and we have $G/N \cong H$.
A similar argument shows that $G/H \cong N$. We only need to notice that $gH = hnH = Hhn = Hn = nH$ because H is normal in G.

(b) To show that $G \cong H \times N$, we can define a map $\psi : H \times N \to G$ by $\psi(h, n) = hn$. For $h \in H, n \in N$, $hnh^{-1}n^{-1} \in H \cap N = \{e\}$. Therefore, elements in H commute with elements in N.
To show that ψ is a homomorphism, note that for any $h_1, h_2 \in H$ and $n_1, n_2 \in N$, we have $\psi((h_1, n_1)(h_2, n_2)) = h_1n_1h_2n_2 = h_1h_2n_1n_2 = \psi(h_1h_2, n_1n_2)$.
The map ψ is surjective because $G = HN$, and if$(h, n) \in \ker(\psi)$, $hn = 1$, and $h = n^{-1} \in H \cap N = \{1\}$. Then $\ker(\psi) = \{(e, e)\}$.
Therefore, ψ is an isomorphism, and $G \cong H \times N$.

□

(5) (10 pts) Let X be a finite G-set such that $|X| \geq 2$. Suppose that the G-action on X is transitive. Show that the fixed point set X_G is empty.

Proof. Assume for contradiction that X_G is not empty, i.e., there exists an $x_0 \in X_G$.

Since the G-action is transitive, for every $x \in X$, there exists a $g_x \in G$ such that $g_x x_0 = x$. However, since $x_0 \in X_G$, we have $g_x x_0 = x_0$. It follows that $x = x_0$ for all $x \in X$.

This contradicts the assumption $|X| \geq 2$. Hence, the assumption that X_G is not empty must be false. Therefore, X_G is empty. □

(6) (20 pts) Let G be a group. Recall that an **inner automorphism** of G is an automorphism of the form

$$i_g : G \to G, \quad a \mapsto gag^{-1}$$

for some $g \in G$. Denote by $\mathrm{Inn}(G)$ the set of all inner automorphisms of G.

(a) Show that $\text{Inn}(G)$ is a normal subgroup of the automorphism group $\text{Aut}(G)$ of G.

(b) Denote by $Z(G)$ the center of G. Show that $G/Z(G) \cong \text{Inn}(G)$. You may use, without proof, the fact that $Z(G)$ is a normal subgroup of G.

Proof. (a) Let G be a group. Define the map $\phi : G \to \text{Aut}(G)$ by $g \mapsto i_g$, where $i_g(x) = gxg^{-1}$ is the conjugation by g. We show that ϕ is a homomorphism. Let $g, h \in G$. Then $i_{gh}(x) = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1} = g(i_h(x))g^{-1} = i_g(i_h(x))$. Note that $\text{Inn}(G) = \phi(G)$. Therefore, $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$.
Let $\phi \in \text{Aut}(G), g \in G$. Then

$$\begin{aligned}
&\phi i_g \phi^{-1}(x) \\
&= \phi(g\phi^{-1}(x)g^{-1}) \\
&= \phi(g)\phi(\phi^{-1}(x))\phi(g^{-1}) \\
&= \phi(g)x(\phi(g))^{-1} \\
&= i_{\phi(g)}(x).
\end{aligned}$$

Therefore, $\text{Inn}(G) \lhd \text{Aut}(G)$.

(b) We continue to use the homomorphism $\phi : G \to \text{Inn}(G)$ defined by $\phi(g) = i_g$.
We claim that the kernel of ϕ is the center $Z(G)$ of G. Indeed, for any $g \in G$, we have $\phi(g) = i_g$ is the identity map on G if and only if $gx = xg$ for all $x \in G$, which is precisely the definition of $Z(G)$.
Therefore, the kernel of ϕ is $Z(G)$, and by the First Isomorphism Theorem, the quotient group $G/\ker(\phi) = G/Z(G)$ is isomorphic to the image of ϕ, which is $\text{Inn}(G)$. □

(7) (15 pts) Let G be a finite group in which every element $g \neq e$ has order 2. Prove that G is isomorphic to a direct product of copies of the cyclic group of order 2.

Proof. First, we show that G is abelian. For any $g, h \in G$, we have $(gh)^2 = e$. But since $g^2 = h^2 = e$, this can be rewritten as $ghgh = e$, and then as $ghg^{-1}h^{-1} = e$. Therefore, $gh = hg$ for any $g, h \in G$, so G is abelian.

Next, we apply the Fundamental Theorem of Finite Abelian Groups, which states that every finite abelian group G is isomorphic to a direct product of cyclic groups of prime power order.

Since every element of G has order 2, the order of each factor must have order 2. Therefore, G is isomorphic to a direct product of copies of the cyclic group of order 2. □